Malfatti's Problem

The problem that has become known as 'Malfatti's Problem' dates back to 1803. It is concerned with finding the three non-overlapping circles in a given triangle which have the greatest total area. Gianfresco Malfatti (1731-1807) stated that the optimal arrangement was one in which each circle touched the two other circles and also two sides of the triangle, as shown in Fig 1; this will be called a *Malfatti arrangement*.

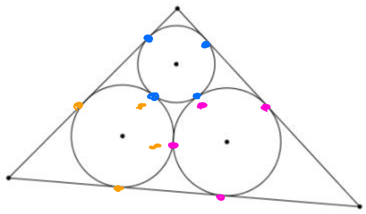


Fig 1

Malfatti believed this provided the solution to the problem. However, over 100 years later it was pointed out that even in the most basic case of the equilateral triangle, the Malfatti arrangement does not provide the maximum area. This is proved below.

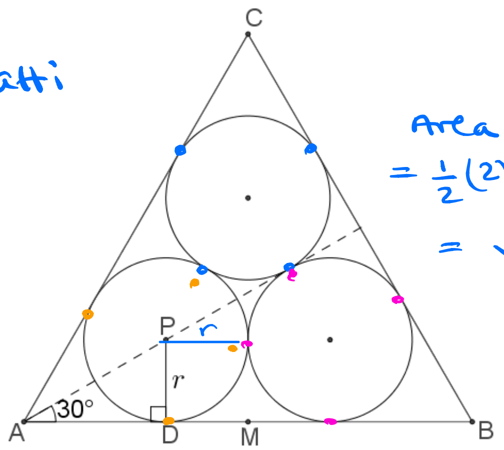


Fig. 2.1

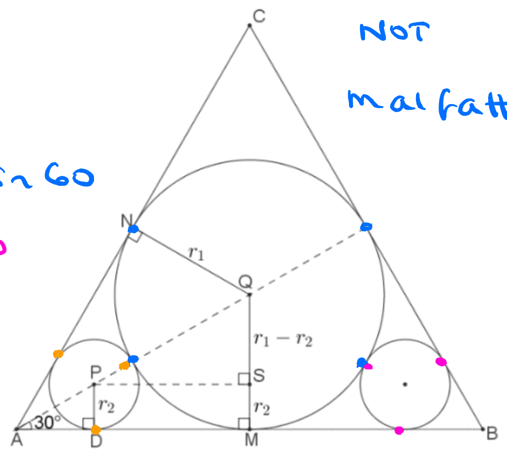


Fig.2.2

Fig. 2.1 shows the Malfatti arrangement for an equilateral triangle, ABC, of side length 2 units, along with the angle bisector at A. M is the midpoint of AB. From triangle ADP it can be seen that $\text{AD} = r\cot 30^\circ$ and so $\text{AM} = r + r\cot 30^\circ$. Since $\text{AM} = 1$ it follows that $r = \dfrac{1}{1+\cot 30^\circ} = \dfrac{\sqrt{3}-1}{2}$ and so the total area of the three circles is $3\pi\left(\dfrac{\sqrt{3}-1}{2}\right)^2 = \dfrac{3\pi}{2}\left(2-\sqrt{3}\right)$ ♠ square units. The circles cover approximately 72.9% of the triangle.

Fig. 2.2 shows an equilateral triangle, ABC, of side length 2 units, along with the angle bisector at A. M and N are the midpoints of AB and AC respectively. From triangle ANQ it can be seen that $\text{QN} = r_1 = \tan 30^\circ$. From triangle QSP it can be seen that $\sin 30^\circ = \dfrac{r_1 - r_2}{r_1 + r_2}$; rearranging this leads to $3r_2 = r_1$ and so $r_2 = \dfrac{\sqrt{3}}{9}$. Therefore the total area of the three circles ♣

is $\frac{11\pi}{27}$ square units. In this arrangement the circles cover approximately 73.9% of the triangle.

In some triangles it is easy to see that the Malfatti arrangement would not provide the maximum area; for example, the Malfatti arrangement for the isosceles triangle shown in Fig. 3.1 clearly produces a smaller total area than the arrangement shown in Fig. 3.2.

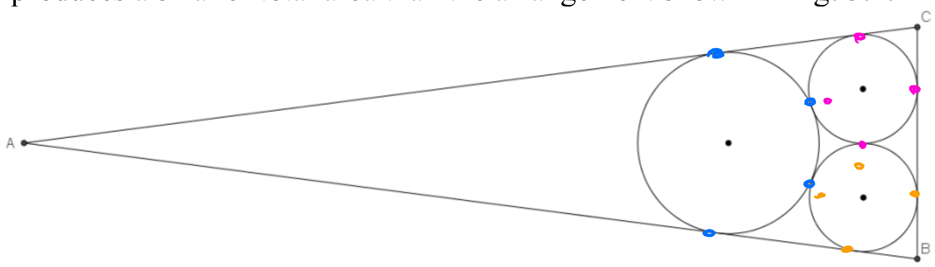


Fig.3.1

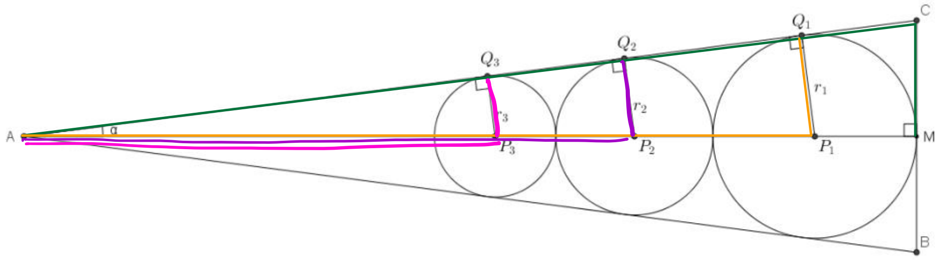


Fig.3.2

To determine this area, start by equating expressions for $\sin\alpha$ in the four right-angled triangles in Fig. 3.2. This gives

$$\frac{\text{CM}}{\text{CA}}=\frac{\text{P}_1\text{Q}_1}{\text{P}_1\text{A}}=\frac{\text{P}_2\text{Q}_2}{\text{P}_2\text{A}}=\frac{\text{P}_3\text{Q}_3}{\text{P}_3\text{A}}$$

which can be written

$$\frac{\text{CM}}{\text{CA}}=\frac{r_1}{\text{AM}-r_1}=\frac{r_2}{\text{AM}-2r_1-r_2}=\frac{r_3}{\text{AM}-2r_1-2r_2-r_3}. \qquad (1)$$

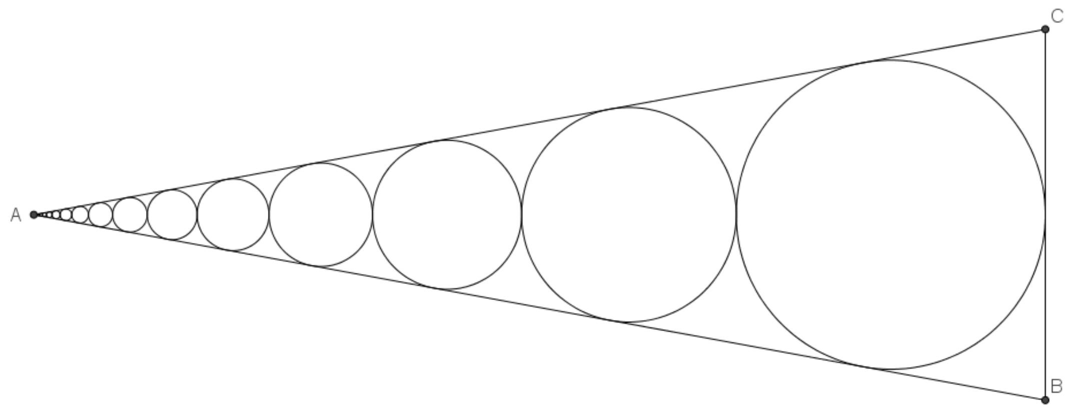
For a given isosceles triangle, the lengths AM, AC and MC will be known and so the three radii can be calculated. For example, in an isosceles triangle with $\text{BC}=22$ units and $\text{AM}=60$ units, (1) becomes

$$\frac{11}{61}=\frac{r_1}{60-r_1}=\frac{r_2}{60-2r_1-r_2}=\frac{r_3}{60-2r_1-2r_2-r_3} \qquad (2)$$

This leads to $r_1=11\times\frac{5}{6}, r_2=11\times\left(\frac{5}{6}\right)^3, r_3=11\times\left(\frac{5}{6}\right)^5$. The area of the isosceles triangle is 660 square units and so the three circles cover approximately 68.6% of this triangle.

To complete the story, in 1967 it was proved that the Malfatti arrangement would not produce the maximum area for any triangle. Malfatti could not have been more wrong!

Possible questions

1	Using the information on line ♠-1 and Fig.2.1, derive the result, given on line ♠, that $r = \dfrac{1}{1+\cot 30^\circ}$. $\mathrm{AM} = r + r\cot 30^\circ$, AM = 1 1 = r(1+ cot 30°) r = 1/(1+ cot 30°)
2	Show that the expressions $3\pi\left(\dfrac{\sqrt{3}-1}{2}\right)^2$ and $\dfrac{3\pi}{2}\left(2-\sqrt{3}\right)$, given on line ♠, are equal.
3	On line ♠+1 it says that the circles in Fig. 2.1 'cover approximately 72.9% of the triangle'. Explain this result.
4	Using the result $\sin 30^\circ = \dfrac{r_1 - r_2}{r_1 + r_2}$ given on line ♣-1, derive the result $r_2 = \dfrac{\sqrt{3}}{9}$ given on line ♣.
5	Explain the result, given on lines ♣ and ♣+1, that 'the total area of the three circles is $\dfrac{11\pi}{27}$ square units.'
6	Comparing equations (1) and (2), explain why $\dfrac{\mathrm{CM}}{\mathrm{CA}} = \dfrac{11}{61}$.
7	Using equation (2), derive the result, given on line ♥, that $(a)\quad r_1 = 11 \times \dfrac{5}{6}$ $(b)\quad r_2 = 11 \times \left(\dfrac{5}{6}\right)^3$
8	The diagram below is an extension of Fig. 3.2 with an infinite number of touching circles. What percentage of the triangle's area is covered by the circles? 

2.

$$3\pi\left(\frac{\sqrt{3}-1}{2}\right)^2 \text{ and } \frac{3\pi}{2}\left(2-\sqrt{3}\right),$$

$$3\pi\left(\frac{\sqrt{3}-1}{2}\right)^2 = \frac{3\pi}{4}(\sqrt{3}-1)(\sqrt{3}-1)$$

$$= \frac{3\pi}{4}(3-2\sqrt{3}+1)$$

$$= \frac{3\pi}{4}(4-2\sqrt{3})$$

$$= \frac{3\pi}{2}(2-\sqrt{3})$$

3.

Area of Δ ABC Fig2.1

$= \frac{1}{2}(2)(2)\sin 60$

$= \sqrt{3}$ (b)

Area of the 3 Circles in Fig 2.1

$= \frac{3\pi}{2}(2-\sqrt{3})$ (a) line ♠

Percentage area covered $= \frac{a}{b} \times 100$

$= \left(\frac{3\pi}{2}(2-\sqrt{3}) \div \sqrt{3}\right) \times 100$

$= 72.9\%$

4.

$$\sin 30^\circ = \frac{r_1 - r_2}{r_1 + r_2}$$

$r_1 = \tan 30$ ①

$$\frac{1}{2} = \frac{r_1 - r_2}{r_1 + r_2}$$

4. Contd.

$$r_1 + r_2 = 2r_1 - 2r_2$$

$$3r_2 = r_1$$

$$r_2 = \frac{r_1}{3}$$

Sub. in ①

$$r_2 = \frac{\tan 30}{3}$$

$$r_2 = \frac{1}{\sqrt{3}} \div 3$$

$$r_2 = \frac{1}{3\sqrt{3}}$$

Rationalising

$$r_2 = \frac{\sqrt{3}}{9}$$

5. Fig 2.2

Area of large circle $= \pi r_1^2$

$$= \pi \left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{\pi}{3} = \frac{9\pi}{27}$$

Area of 2 small circles $= 2 \times \pi \times r_2^2$

$$= 2 \times \pi \times \left(\frac{\sqrt{3}}{9}\right)^2$$

$$= 2\pi \times \frac{3^1}{3 \times 27}$$

$$= \frac{2\pi}{27}$$

Total area = large + 2 small

$$= \frac{9\pi}{27} + \frac{2\pi}{27} = \frac{11\pi}{27}$$

6. $CM = \frac{1}{2}BC = \frac{1}{2}(22) = 11$

$CA = \sqrt{60^2 + 11^2}$
$= 61$

$\frac{CM}{CA} = \frac{11}{61}$

7.

$$\frac{11}{61} = \frac{r_1}{60 - r_1} = \frac{r_2}{60 - 2r_1 - r_2} = \frac{r_3}{60 - 2r_1 - 2r_2 - r_3}$$

$$\frac{11}{61} = \frac{r_1}{60 - r_1}$$

$11(60 - r_1) = 61 r_1$

$11 \times 60 - 11 r_1 = 61 r_1$

$11 \times 60 = 72 r_1$

$r_1 = \frac{11 \times 60^{5}}{72\ 6}$

$r_1 = 11 \times \left(\frac{5}{6}\right)$

$$\frac{11}{61} = \frac{r_2}{60 - 2r_1 - r_2}$$

$11(60 - 2r_1 - r_2) = 61 r_2$ Sub. $r_1 = 11 \times 5/6$

$11\left(60 - 2 \times 11 \times \left(\frac{5}{6}\right)\right) = 72 r_2$ $\div 72$

$11\left(\frac{60}{72} - \frac{2 \times 11}{72} \times \frac{5}{6}\right) = r_2$

$11\left(\frac{5}{6} - \frac{2 \times 11}{72\,36} \times \frac{5}{6}\right) = r_2$

$11 \times \frac{5}{6}\left(1 - \frac{11}{36}\right) = r_2$

$r_2 = 11 \times \frac{5}{6} \times \left(\frac{36 - 11}{36}\right)$

$r_2 = 11 \times \frac{5}{6} \times \frac{25}{36}$

$r_2 = 11 \times \frac{5}{6} \times \left(\frac{5}{6}\right)^2$

$r_2 = 11 \times \left(\frac{5}{6}\right)^3$

8.

$$r_1 = 11 \times \frac{5}{6}, r_2 = 11 \times \left(\frac{5}{6}\right)^3, r_3 = 11 \times \left(\frac{5}{6}\right)^5.$$

Assume same dimensions as Fig 3.2

Sum of area of circles $= \pi(r_1)^2 + \pi(r_2)^2 + \pi(r_3)^2 + \ldots$

$= \pi \times \left(11 \times \frac{5}{6}\right)^2 + \pi \times \left(11 \times \left(\frac{5}{6}\right)^3\right)^2 + \pi \times \left(11 \times \left(\frac{5}{6}\right)^5\right)^2 + \ldots$

$= \pi \times 11^2 \times \left(\frac{5}{6}\right)^2 + \pi \times 11^2 \times \left(\frac{5}{6}\right)^6 + \pi \times 11^2 \times \left(\frac{5}{6}\right)^{10} + \ldots$

Sum of a G.P.

$S_\infty = \frac{a}{1-r}$

$a = \pi \times 11^2 \times \left(\frac{5}{6}\right)^2$

$r = \left(\frac{5}{6}\right)^4$

$= \frac{\pi \times 11^2 \times \left(\frac{5}{6}\right)^2}{1 - (5/6)^4}$

$= 509.9$

Area of Δ = 660 Line ❤

Percentage Area Covered

$= \frac{509.9}{660} \times 100$

$= \underline{\underline{77.3\%}}$